

MAS212 Linear Algebra I Exercises 3 (with solutions)

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Assessed submission deadline 5pm on Friday 20th October 2006 in the blue box on the ground floor. Your submission must have your FULL NAME and STUDENT NUMBER PRINTED CLEARLY at the top and separate sheets must be stapled together. Marks out of 100 are shown in square brackets after each question number.

1. [20] In the vector space $\mathbb{R}^3$, determine whether each of the vectors $u = (1, 2, 0)$ and $v = (0, 1, 2)$ is in the vector subspace spanned by $(1, 2, 3)$ and $(2, 1, 0)$.

Solution

For u , try to solve $(1, 2, 0) = a(1, 2, 3) + b(2, 1, 0)$ for $a, b \in \mathbb{R}$. Split the vector equation into three coupled scalar equations to give

$$\begin{aligned}1 &= a + 2b, \\2 &= 2a + b, \\0 &= 3a.\end{aligned}$$

The last equation implies that $a = 0$. Then the first two equations become $1 = 2b$, $2 = b$, which are clearly inconsistent (since $1 \neq 4$), so there are no solutions. Therefore, $u \notin \langle (1, 2, 3), (2, 1, 0) \rangle$.

For v , try to solve $(0, 1, 2) = a(1, 2, 3) + b(2, 1, 0)$ for $a, b \in \mathbb{R}$. Splitting the vector equation into three coupled scalar equations gives

$$\begin{aligned}0 &= a + 2b, \\1 &= 2a + b, \\2 &= 3a.\end{aligned}$$

The last equation gives $a = \frac{2}{3}$. Then the first equation gives $b = -\frac{a}{2} = -\frac{1}{3}$. This solution is

consistent with the second equation since $1 = 2\left(\frac{2}{3}\right) + \left(-\frac{1}{3}\right)$. Therefore,

$v \in \langle (1, 2, 3), (2, 1, 0) \rangle$.

Maple Solution

2. [20] (a) Let $v_1 = (1, 2)$, $v_2 = (2, 1) \in \mathbb{R}^2$. Show that $\{v_1, v_2\}$ is a linearly independent set of vectors.

(b) Let $v_1 = (1, 0, 2)$, $v_2 = (2, 1, -1)$, $v_3 = (1, 1, 1) \in \mathbb{R}^3$. Show that $\{v_1, v_2, v_3\}$ is a linearly independent set of vectors.

(c) Let $v_1 = (1, 0, -1, -1)$, $v_2 = (0, 0, 1, -1)$, $v_3 = (0, -1, 0, 0)$, $v_4 = (-2, 2, 1, 3) \in \mathbb{R}^4$. Show that $\{v_1, v_2, v_3, v_4\}$ is a linearly dependent set of vectors.

Solution

(a) To determine whether the vectors are linearly independent we need to solve $a(1, 2) + b(2, 1) = 0$ for $a, b \in \mathbb{R}$. This vector equation is equivalent to the two scalar equations

$$\begin{aligned}a + 2b &= 0, \quad (1) \\2a + b &= 0. \quad (2)\end{aligned}$$

(1) $\Rightarrow a = -2b$. Substituting for a in (2) gives $-4b + b = -3b = 0$, hence the only solution is $a = b = 0$. Therefore the vectors are linearly independent.

(b) Solve $a(1, 0, 2) + b(2, 1, -1) + c(1, 1, 1) = 0$ for $a, b, c \in \mathbb{R}$, i.e.

$$\begin{aligned}a + 2b + c &= 0, \quad (1) \\b + c &= 0, \quad (2) \\2a - b + c &= 0. \quad (3)\end{aligned}$$

(2) $\Rightarrow b = -c$. Substituting for b in (1) and (3) gives $a - 2c + c = a - c = 0 \Rightarrow a = c$ and $2a + c + c = 2a + 2c = 0 \Rightarrow a = -c$. Hence the only solution is $a = b = c = 0$. Therefore the vectors are linearly independent.

(c) Solve $a(1, 0, -1, -1) + b(0, 0, 1, -1) + c(0, -1, 0, 0) + d(-2, 2, 1, 3) = 0$ for $a, b, c, d \in \mathbb{R}$, i.e.

$$\begin{aligned} a - 2d &= 0, & (1) \\ -c + 2d &= 0, & (2) \\ -a + b + d &= 0, & (3) \\ -a - b + 3d &= 0. & (4) \end{aligned}$$

(1) $\Rightarrow a = 2d$ and (2) $\Rightarrow c = 2d$. Substituting for a in (3) and (4) gives $b - d = 0$, $-b + d = 0$. These two equations are equivalent and imply that $b = d$, so the solution is $a = 2d$, $b = d$, $c = 2d$ for any $d \in \mathbb{R}$. Clearly $a = b = c = d = 0$ is a solution, but this solution is not unique and there are nonzero solutions for any nonzero $d \in \mathbb{R}$. Therefore the vectors are linearly dependent.

► Maple Solution

3. [20] In the vector space $\mathbb{R}^3$, (a) find a maximal (i.e. as large as possible) linearly independent subset of the set of vectors $\{(1, -2, 1), (2, 1, 2), (1, -1, 1), (1, 1, 1)\}$, and (b) extend it to a linearly independent set of three vectors and show that this forms a basis for $\mathbb{R}^3$.

▼ Solution

(a) There cannot be more than 3 linearly independent vectors in $\mathbb{R}^3$, so we can discard at least one. By inspection, $(1, -1, 1) + (1, 1, 1) = (2, 0, 2)$ and $(1, 1, 1) - (1, -1, 1) = (0, 2, 0)$. From these two vectors we can construct the first two vectors in the set as

$$(1, -2, 1) = \frac{1}{2}(2, 0, 2) - (0, 2, 0) \text{ and } (2, 1, 2) = (2, 0, 2) + \frac{1}{2}(0, 2, 0). \text{ Moreover,}$$

$\{(1, -1, 1), (1, 1, 1)\}$ is clearly a linearly independent set since one vector is not a scalar multiple of the other, and since the other two vectors are linear combinations of these two vectors, this is a maximal linearly independent set.

(b) Try including the first standard basis vector for $\mathbb{R}^3$ to give $\{(1, -1, 1), (1, 1, 1), (1, 0, 0)\}$. To prove that this set is linearly independent solve $a(1, -1, 1) + b(1, 1, 1) + c(1, 0, 0) = 0$ for $a, b, c \in \mathbb{R}$, i.e.

$$\begin{aligned} a + b + c &= 0, & (1) \\ -a + b &= 0, & (2) \\ a + b &= 0. & (3) \end{aligned}$$

(2) + (3) $\Rightarrow 2b = 0 \Rightarrow b = 0$ and (3) - (2) $\Rightarrow 2a = 0 \Rightarrow a = 0$. Then (1) $\Rightarrow c = 0$. Hence $a = b = c = 0$ uniquely and so the vectors are linearly independent. To determine whether they also span $\mathbb{R}^3$ we must solve $a(1, -1, 1) + b(1, 1, 1) + c(1, 0, 0) = (x, y, z) \in \mathbb{R}^3$ for $a, b, c \in \mathbb{R}$, i.e.

$$\begin{aligned} a + b + c &= x, & (1) \\ -a + b &= y, & (2) \\ a + b &= z. & (3) \end{aligned}$$

$$(2) + (3) \Rightarrow 2b = y + z \Rightarrow b = \frac{y+z}{2} \text{ and } (3) - (2) \Rightarrow 2a = z - y \Rightarrow a = \frac{z-y}{2}. \text{ Then}$$

(1) $\Rightarrow c = x - (a + b) = x - z$. Hence there are solutions for $a, b, c \in \mathbb{R}$ for any $x, y, z \in \mathbb{R}$ and so the set of vectors spans $\mathbb{R}^3$. Therefore $\{(1, -1, 1), (1, 1, 1), (1, 0, 0)\}$ forms a basis for $\mathbb{R}^3$.

► Maple Solution

4. [20] (a) Let $v_1 = (1, 1)$ and $v_2 = (1, 2)$. Show from first principles that $\{v_1, v_2\}$ forms a basis for $\mathbb{R}^2$.

(b) Let $v_1 = (1, 2)$, $v_2 = (2, 1)$ and $v_3 = (1, 1)$. Show that $\{v_1, v_2, v_3\}$ is a spanning set for $\mathbb{R}^2$. Is it also a basis?

(c) Let $v_1 = (0, 1, 0)$, $v_2 = (1, 2, 1)$, $v_3 = (0, -1, 1)$ and $v_4 = (1, 0, -1)$. Show that $\{v_1, v_2, v_3, v_4\}$ is a spanning set for $\mathbb{R}^3$. Is it also a basis?

Solution

(a) A basis is a linearly independent spanning set. Suppose $a(1, 1) + b(1, 2) = 0$. This corresponds to the two scalar equations $a + b = 0$, $a + 2b = 0 \Rightarrow a = 0, b = 0$. Therefore v_1, v_2 are linearly independent. Suppose $(x, y) = a(1, 1) + b(1, 2)$. This corresponds to the two scalar equations $x = a + b$, $y = a + 2b \Rightarrow a = 2x - y$, $b = y - x$. Therefore any $(x, y) \in \mathbb{R}^2$ can be expressed as $(x, y) = (2x - y)(1, 1) + (y - x)(1, 2)$, so $\{v_1, v_2\}$ spans $\mathbb{R}^2$ and therefore forms a basis for $\mathbb{R}^2$.

(b) Suppose $(x, y) = a(1, 2) + b(2, 1) + c(1, 1) \Rightarrow$

$$x = a + 2b + c, \quad (1)$$

$$y = 2a + b + c. \quad (2)$$

We have only 2 equations, so we can solve for only 2 unknowns, say a, b , so regard c as a parameter.

$2 \cdot (1) - (2) \Rightarrow 2x - y = 3b + c \Rightarrow 3b = 2x - y - c \Rightarrow b = \frac{2x - y - c}{3}$. Substituting this into

$(1) \Rightarrow a = x - 2b - c = x - \frac{2}{3}(2x - y - c) - c = \frac{-x + 2y - c}{3}$. This solution holds $\forall x, y, c \in \mathbb{R}$, therefore $\{v_1, v_2, v_3\}$ is a spanning set for $\mathbb{R}^2$. However, it is not a basis because it cannot be linearly independent. To prove this, set $x = y = 0$ in the above equations to give $a = \frac{-c}{3}$,

$b = \frac{-c}{3} \quad \forall c \in \mathbb{R}$, which is not unique and gives nonzero solutions for any $c \neq 0$.

(c) Suppose $(x, y, z) = a(0, 1, 0) + b(1, 2, 1) + c(0, -1, 1) + d(1, 0, -1) \Rightarrow$

$$x = b + d, \quad (1)$$

$$y = a + 2b - c, \quad (2)$$

$$z = b + c - d. \quad (3)$$

Solving for a, b, c regarding d as a parameter, $(1) \Rightarrow b = x - d$. Then $(3) \Rightarrow$

$c = z - b + d = z - x + 2d$ and $(2) \Rightarrow$

$a = y - 2b + c = y - 2x + 2d + z - x + 2d = y - 3x + 4d + z$. This solution holds

$\forall x, y, z, d \in \mathbb{R}$, therefore $\{v_1, v_2, v_3, v_4\}$ is a spanning set for $\mathbb{R}^3$. However, it is not a basis

because it cannot be linearly independent. To prove this, set $x = y = z = 0$ in the above equations to give $a = 4d, b = -d, c = 2d \quad \forall d \in \mathbb{R}$, which is not unique and gives nonzero solutions for any $d \neq 0$.

Maple Solution

5. [20] Find a basis for U , where:

(a) $U = \{(x, y, z) \mid x, y, z \in \mathbb{R}, x + 2y - z = 0\}$ is a vector subspace of $\mathbb{R}^3$;

(b) $U = \{(w, x, y, z) \mid w, x, y, z \in \mathbb{R}, x - y = 0, w + z = 0\}$ is a vector subspace of $\mathbb{R}^4$.

Solution

(a) Solving the constraint for x (say) gives $x = -2y + z$. Therefore,

$U = \{(-2y + z, y, z) \mid y, z \in \mathbb{R}\} = \{y(-2, 1, 0) + z(1, 0, 1) \mid y, z \in \mathbb{R}\}$, so $\{(-2, 1, 0), (1, 0, 1)\}$ is a spanning set for U . It can easily be proved to be a linearly independent set (since one vector is not a scalar multiple of the other), therefore it is a basis for U .

(b) Solving the constraints for x, w (say) gives $x = y, w = -z$. Therefore,

$U = \{(-z, y, y, z) \mid y, z \in \mathbb{R}\} = \{y(0, 1, 1, 0) + z(-1, 0, 0, 1) \mid y, z \in \mathbb{R}\}$, so $\{(0, 1, 1, 0), (-1, 0, 0, 1)\}$ is a spanning set for U . It can easily be proved to be a linearly

└ independent set (since one vector is not a scalar multiple of the other), therefore it is a basis for U .